Lecture 10 : 9-10-18

§ Higher order equation :

Def: • n-th order linear equation :

$$(*) \cdots\cdots \boxed{y^{(n)} + p_{n-1}(t)\,y^{(n-1)} + \cdots\cdots + p_1(t)\,y^{(1)} + p_0(t)\,y = r(t)}$$

where $p_i(t)$, $r(t)$ continuous function on $I = (a,b)$.

IVP : $y(t_0) = x_0, \ldots\ldots, y^{(n-1)}(t_0) = x_{n-1}$

• It is called homogeneous if $r(t) \equiv 0$

Thm: (Existence and uniqueness)
For the IVP problem, their $\exists!$ solution $y(t)$ defined on the whole I.

Prop: (superposition principle)
Let $y_1(t)$, $y_2(t)$ solution to

$$y_i^{(n)} + p_{n-1}(t)\,y_i^{(n-1)} + \cdots + p_1(t)\,y_i' + p_0(t)\,y_i = r_i(t)$$

then $y = c_1 y_1 + c_2 y_2$ for constant c_1, c_2
solution to :

$$y^{(n)} + p_{n-1}(t)\,y^{(n-1)} + \cdots\cdots + p_0(t)\,y = c_1 r_1 + c_2 r_2$$

• In particular, if both y_1, y_2 solution to homogeneous equation (i.e. $r_i \equiv 0$)

$\Rightarrow$ y is again a solution to homogeneous eqt.

Pf: $c_1\left(y_1^{(n)} + p_{n-1}(t)\, y_1^{(n-1)} + \cdots + p_0(t)\, y_1\right) = c_1\, r_1(t)$

$+\ c_2\left(y_2^{(n)} + p_{n-1}(t)\, y_2^{(n-1)} + \cdots + p_0(t)\, y_2\right) = c_2\, r_2(t)$

$= \ y^{(n)} + p_{n-1}(t)\, y^{(n-1)} + \cdots + p_0(t)\, y = c_1 r_1 + c_2 r_2$ $\square$

Cor: If we consider homogeneous eqt:

$(**) \ \ldots \ \boxed{y^{(n)} + p_{n-1}(t)\, y^{(n-1)} + \cdots + p_0(t)\, y = 0}$

Then the set of solution

$$\mathcal{S} := \{\, y \mid y \text{ satisfy } (**) \,\}$$

is a $\mathbb{R}$ vector space.

meaning: we have operation

$+ : \mathcal{S} \times \mathcal{S} \longrightarrow \mathcal{S}$

$\cdot : \mathbb{R} \times \mathcal{S} \longrightarrow \mathcal{S}$ like what we have for vectors.

Abstractly: A $\mathbb{R}$-vector space $\mathcal{V}$ is

$$+ : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$$
$$\cdot : \mathbb{R} \times \mathcal{V} \longrightarrow \mathcal{V}$$

They should satisfy:

1) (Exist 0) There exist $0 \in \mathcal{V}$ s.t. $0 + v = v + 0 = v$.

2) (commutative of +) $v_1 + v_2 = v_2 + v_1$

3) (associativity of +) $v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$

4) (existence of inverse of +) For any $v \in \mathcal{V}$, there is an element w s.t. $v + w = 0$

5) (associativity of scalar $\cdot$) $r(sv) = (r \cdot s)v$ for $r, s \in \mathbb{R}$, $v \in \mathcal{V}$

6) (distributive of scalar $\cdot$) $(r+s)v = r v + s \cdot v$ $r, s \in \mathbb{R}$, $v \in \mathcal{V}$

7) (distributive of vector +) $r(v_1 + v_2) = r v_1 + r v_2$ $r \in \mathbb{R}$, $v_1, v_2 \in \mathcal{V}$

8) (1 acts as identity) $1 \cdot v = v$ for $v \in \mathcal{V}$.

Def: (linear independence)

For $y_1, \dots, y_k \in \mathcal{S}$, i.e. solution to (**) is said to be linearly independent if they satisfy:

$$\begin{cases} \text{For any } c_1, \dots, c_k \in \mathbb{R} \text{ with} \\ \quad c_1 y_1 + \dots + c_k y_k = 0 \\ \text{then we must have } c_1 = 0 \dots = c_k \end{cases}$$

Rk: If $y_1, \dots, y_k$ is linearily dependent

$\Rightarrow \exists\, c_1, \dots, c_k$ NOT all zero s.t.

$$c_1 y_1 + \dots + c_k y_k = 0$$

omitting this term!

Say $c_j \neq 0$:

$$y_j = -\frac{1}{c_j}\left(c_1 y_1 + \dots \widehat{c_j y_j} + \dots + c_k y_k\right)$$

i.e. y_j can be expressed as a linear combination of other y_i's.

E.g.: Assume we have $y_1 = 1$, $y_2 = t$, $y_3 = t^2$ in $\mathcal{S}$ then they are linearily independent!

To see that: if $c_1 y_1 + c_2 y_2 + c_3 y_3 = 0$

Let $t=0$: $c_1 = 0$

Let $t=1$: $c_1 + c_2 + c_3 = 0$

Let $t=-1$: $c_1 - c_2 + c_3 = 0$

$\Rightarrow c_1 = 0 = c_2 = 0 = c_3$.

Def: • For $y_1, \dots, y_k \in \mathcal{S}$, we let

$$\mathrm{Span}(y_1, \dots, y_k) := \left\{ y \mid \begin{array}{l} y = c_1 y_1 + \dots + c_k y_k \\ c_i \in \mathbb{R} \end{array} \right\}$$

the be the subset called the Span of $y_1, \dots, y_k$

• $y_1, \dots, y_k$ is called basis of $\mathcal{S}$

1) $\mathrm{Span}(y_1, \dots, y_k) = \mathcal{S}$

2) linearily independent.

Def: If $y_1, \dots, y_n$ are solution to (**), we define their Wronskin $W(y_1, \dots, y_n)(t)$ as

$$W(y_1,\dots,y_n)(t) := \det \begin{pmatrix} y_1 & \cdots & y_n \\ y_1^{(1)}, & \cdots & , y_n^{(1)} \\ \vdots & & \\ y_1^{(n-1)}, & \cdots & , y_n^{(n-1)} \end{pmatrix}$$

Thm: Let $y_1,\dots,y_n$ Solution to $(**)$, then TFAE:

1) $y_1,\dots,y_n$ **basis for** $\mathcal{S}$

2) $W(y_1,\dots,y_n)(t_0) \neq 0$

Pf: 2) $\Rightarrow$ 1)

Assume $c_1y_1 + \dots + c_ny_n = 0$, then evaluation at t_0 we have

then:
$$\begin{pmatrix} y_1(t_0) & \cdots & y_n(t_0) \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t_0) & \cdots & y_n^{(n-1)}(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

then since the matrix is invertible

$\Rightarrow \quad c_1 = 0 \dots\dots = c_n = 0$

1) $\Rightarrow$ 2) Assume that we have $y_1, \dots, y_n$ linearily independent.

$\mathrm{Span}(y_1, \ldots, y_n) = \mathcal{S}$:

then let $y \in \mathcal{S}$, we want to find

$c_1, \ldots, c_n \in \mathbb{R}$ s.t $c_1 y_1 + \ldots + c_n y_n = y$

consider the initial value:

$$x_0 = y(t_0), \quad \ldots\ldots, \quad x_{n-1} = y^{(n-1)}(t_0)$$

by $W(y_1, \ldots, y_n)(t_0) \neq 0$:

$$\begin{pmatrix} y_1(t_0) & \cdots & y_n(t_0) \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t_0) & \cdots & y_n^{(n-1)}(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} x_0 \\ \vdots \\ x_{n-1} \end{pmatrix}$$

has an unique sol $c_1, \ldots, c_n$.

1) $\Rightarrow$ 2) if $y_1, \ldots, y_n$ basis for $\mathcal{S}$,

we consider solution to initial value problem

$$\begin{matrix} z_0(t_0) = 1, & z_1^{(1)}(t_0) = 0, & \ldots, & z_1^{(n-1)}(t_0) = 0 \\ \vdots & & \ddots & \vdots \\ z_{n-1}(t_0) = 0, & z_{n-1}^{(1)}(t_0) = 0, & \ldots & z_{n-1}^{(n-1)}(t_0) = 1 \end{matrix}$$

$\Rightarrow$ For each z_i we have

$$c_{1i} y_1 + \ldots + c_{ni} y_n = z_i$$

and hence taking the evaluation of the derivatives at t_0 we get:

$$\begin{pmatrix} y_1(t_0) & \cdots & y_n(t_0) \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t_0) & \cdots, & y_n^{(n-1)}(t_0) \end{pmatrix} \begin{pmatrix} c_{10} & & c_{1,n-1} \\ \vdots & \cdots & \vdots \\ c_{n0} & & c_{n,n-1} \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$\Rightarrow \qquad W(y_1, \ldots, y_n)(t_0) \neq 0 .$$

Thm: Consider (**), we let $y_1, \ldots, y_n$ be solution to (**), and if we let

$$W(t) := W(y_1, \ldots, y_n)(t),$$

then:

$$W'(t) + p_{n-1}(t)\, W(t) = 0$$

$$\Rightarrow \qquad W(t) = c\, e^{-\int p_{n-1}(t)\,dt} \quad \text{for } c \text{ independent}$$

of t. (only depend on $y_1, \ldots, y_n$)

Claim:

Pf: $(n=3)$ we have

$$\frac{d}{dt}\left[\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\right] = \det\begin{pmatrix} a_{11}' & a_{12}' & a_{13}' \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$+ \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}' & a_{22}' & a_{23}' \\ a_{31} & a_{32} & a_{33} \end{pmatrix} + \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31}' & a_{32}' & a_{33}' \end{pmatrix}$$

Case $n=2$:

$$\frac{d}{dt}\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix} = \frac{d}{dt}\left(a_{11}a_{22} - a_{12}a_{21}\right)$$

$$= a_{11}'a_{22} - a_{12}'a_{21} + a_{11}a_{22}' - a_{12}a_{21}'$$

$$= \det\begin{pmatrix} a_{11}' & a_{12}' \\ a_{21} & a_{22}\end{pmatrix} + \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21}' & a_{22}'\end{pmatrix}$$

Induction Step: use $n=2$ to prove $n=3$

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix} = a_{11}\det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33}\end{pmatrix} - a_{12}\det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33}\end{pmatrix} + a_{13}\det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32}\end{pmatrix}$$

$$\frac{d}{dt}\det\begin{pmatrix} a_{11} & & \\ & \ddots & \\ & & a_{33}\end{pmatrix} = \boxed{a_{11}'\det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33}\end{pmatrix} - a_{12}'\det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33}\end{pmatrix} + a_{13}'\det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32}\end{pmatrix}}$$
$$+ \boxed{a_{11}\det\begin{pmatrix} a_{22}' & a_{23}' \\ a_{32} & a_{33}\end{pmatrix} - a_{12}\det\begin{pmatrix} a_{21}' & a_{23}' \\ a_{31} & a_{33}\end{pmatrix} + a_{13}\det\begin{pmatrix} a_{21}' & a_{22}' \\ a_{31} & a_{32}\end{pmatrix}}$$
$$+ \boxed{a_{11}\det\begin{pmatrix} a_{22} & a_{23} \\ a_{32}' & a_{33}'\end{pmatrix} - a_{12}\det\begin{pmatrix} a_{21} & a_{23} \\ a_{31}' & a_{33}'\end{pmatrix} + a_{13}\det\begin{pmatrix} a_{21} & a_{22} \\ a_{31}' & a_{32}'\end{pmatrix}}$$

$$= \det\begin{pmatrix} a_{11}' & a_{12}' & a_{13}' \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix} + \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}' & a_{22}' & a_{23}' \\ a_{31} & a_{32} & a_{33}\end{pmatrix} + \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31}' & a_{32}' & a_{33}'\end{pmatrix}$$

In general true for $n \times n$ matrix:

$$\frac{d}{dt}\det\begin{pmatrix} a_{11} & & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & & a_{nn}\end{pmatrix} = \det\begin{pmatrix} a_{11}' & \cdots & a_{1n}' \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn}\end{pmatrix} +$$

$$\det\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21}' & \cdots & a_{21}' \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} + \cdots + \det\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1}' & \cdots & a_{nn}' \end{pmatrix}$$

Back to Abel's thm:

$$\frac{d}{dt}\det\begin{pmatrix} y_1 & \cdots & y_n \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} = \det\begin{pmatrix} y_1^{(1)} & \cdots & y_1^{(1)} \\ y_1^{(1)} & \cdots & y_1^{(1)} \\ \vdots & & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}$$

$$+ \cdots + \det\begin{pmatrix} y_1 & \cdots & y_n \\ \vdots & \ddots & \vdots \\ y_1^{(n)} & \cdots & y_n^{(n)} \end{pmatrix}$$

$$\det\begin{pmatrix} y_1 & \cdots & y_n \\ y_1' & & y_n' \\ \vdots & \ddots & \vdots \\ -p_{n-1}y_1^{(n-1)} - \cdots - p_0 y_1 & \cdots & -p_{n-1}y_n^{(n-1)} - \cdots - p_0 y_n \end{pmatrix}$$

$$= -p_{n-1}\det\begin{pmatrix} y_1 & \cdots & y_n \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} = -p_{n-1}(t)\,W(t)$$

$\square$

Cor: If $W(y_1,\dots,y_n)(t_0) \neq 0 \Rightarrow W(y_1,\dots,y_n)(t) \neq 0$ for all $t \in I$.